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A PRELIMINARY CALIBRATION OF THE HERTZSPRUNG-RUSSELL DIAGRAM IN TERMS OF MASS AND AGE FOR POPULATION I MAIN-SEQUENCE STARS OF 1.0-1.6 SOLAR MASSES

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Synopsis

Three series of evolutionary tracks are derived for stars in the mass range 1.0–1.6 solar masses with chemical compositions given by $Z = 0.03$, $X = 0.60$ and 0.70, respectively. The evolution is followed from the zero-age line until the secondary contraction begins. By means of these tracks the corresponding region of the Hertzsprung-Russell diagram is calibrated in terms of mass and age. Finally the accuracy of the calibration is discussed; it is concluded that mass and age determination for single main-sequence stars is considerably more difficult for spectral types later than about F0 than for stars of earlier types, and that this method can not be applied at all for spectral types later than about G0.

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1. Introduction

Calibrations of the main-sequence band in terms of age and mass have been given by KELSALL and STRÖMGREN (1966) for B and A stars for several Population I chemical compositions. STRÖMGREN (1963) has derived ages of early type stars by means of such calibrations. For the later spectral types the calculation of the evolutionary tracks, on which age calibrations have to be based, is somewhat more complicated than for the early types. This is due to the fact that the structure of the stellar envelopes, especially of the ionization zones, has a considerable influence on the properties of the stellar models for lower effective temperatures, in contrast to the upper main-sequence, where the influence of the outer convection and ionization zones is negligible.

BAKER (1963) has calculated envelope models for stars with a mass in the range $0.00 \leq \log M/M_{\odot} \leq 0.20$ for the two Population I chemical compositions $Z = 0.03$, $X = 0.60$ and 0.70, respectively. For the mass values $\log M/M_{\odot} = 0.00$, 0.05, 0.10, 0.15, and 0.20, and both compositions, BAKER constructed two sets of envelope models, for $\alpha = 1$ and $\alpha = 2$ respectively, α being the mixing length in units of the local pressure scale height. Each set consists of 120 models in a network of $\log R/R_{\odot}$ and $\log L/L_{\odot}$ values with $\Delta \log R/R_{\odot} = 0.02$ and $\Delta \log L/L_{\odot} = 0.05$. The results for a family of models of the same chemical composition and the same α value are given in 5 tables, one for each mass value. One table supplies the values of $\log T$, $\log P$, $\log \varrho$, and $r/R_{\odot}$ at a certain point in the envelope model, for instance at a relative mass equal to 0.60.

The BAKER-tables form a basis for subsequent calculations of evolutionary tracks. Data found by interpolation in the tables fix the conditions for the boundary-value problems that have to be solved, i.e. the starting values for the inwards-running integrations. In the present computations, which were performed in the summer of 1965, we have used several of the BAKER-tables in the manner just described. For the chemical composition

$X = 0.70$ and $Z = 0.03$ we have considered both $\alpha = 1.0$ and $\alpha = 2.0$, while we have derived evolutionary tracks only for the case $\alpha = 2.0$ for the chemical composition given by $X = 0.60$ and $Z = 0.03$.

In Section 2 the procedure for model construction is described, and in Section 3 the main properties of the derived evolutionary tracks are analysed. The age calibrations based on these tracks are discussed in Section 4.

2. Model Construction

The evolutionary sequences are derived by means of a computer programme, which is essentially the same as the one described by Reiz and Petersen (1964), in particular the fitting procedure and the handling of the opacity tables are not altered. The physical variables $ln\,P$, $ln\,T$, $q = M(r)/M$, and $f = L(r)/L$ are obtained as function of $x = r/R$ by means of integration of the four basic differential equations from the bottom of the outer zone ($q = 0.60$) to a pre-assigned fitting-point, and from the centre to the fitting-point. The method of integration and the fitting procedure are precisely those described by Reiz and Petersen.

In computing the energy generation rate, the formulae given by Reeves (1965) for the *pp* and *CN* reactions are used directly, while the energy release due to changes in the stellar structure with time is neglected, as this term is known to have a negligible influence on the evolution of main-sequence models. The opacity values are determined by interpolation in tables giving $\log \varkappa$ as function of $\log T$ and $\log \varrho$ for a specific chemical composition defined by means of the relative contents of hydrogen (X) and heavy elements (Z). These tables were computed at the Institute for Space Studies, New York, by means of the procedure given by Cox (1965), and made available to the Copenhagen Observatory by B. Strömgren. Besides bound-free and free-free absorption, and electron scattering, which were included in the tables used by Baker (1963), also bound-bound absorption (lines) is taken into account in the opacity tables used in the central parts of the present models.

In the equation of state non-relativistic, partial degeneracy is taken into account, using a Chebyshev expansion for the ratio of the Fermi-Dirac functions, as described by Reiz (1965). The corrections to the perfect gas law were, however, found to be small in all cases, amounting to at most a few per cent.

The values of the physical and astronomical constants are taken from

ALLEN (1955). All absolute magnitudes are therefore derived using $M_{\mathrm{bol},\odot} = 4.62$. When comparison is made with results based on the more recent value $M_{\mathrm{bol},\odot} = 4.72$ (ALLEN 1963) the difference in magnitude scale must be remembered.

3. Results of Calculations

For the original chemical composition $X = 0.70$ and $Z = 0.03$ two series of evolutionary tracks are derived with different assumptions for α. In one series the inwards-running integrations are started by means of second order interpolation in the BAKER-tables nos. 12–15 (for $\alpha = 1$), and in the other the tables nos. 26–30 ($\alpha = 2$) are used. Figure 1 shows these tracks in the ($\log T_e$, M_{bol}) diagram; the main characteristics of the models, together with similar information about a sequence of tracks derived for the original composition $X = 0.60$ and $Z = 0.03$ are summarized in Tables 1 and 2.

From Figure 1 it is seen that a gradual change in the evolutionary tracks occurs in the mass range covered by the present models. For the smaller mass values we find models with radiative cores, while the models of about 1.6 solar masses possess a convective core containing 12–16 per cent of the total mass in the homogeneous phase. Also the structure of the envelopes are rather different in the two cases. The heavier models have envelopes, in which the convective zones due to ionization of hydrogen and helium, are so shallow that they only have negligible influence on the structure of the whole star, whereas these zones for the models with about one solar mass have a considerable influence on the total radii of the models, and therefore also on the effective temperatures. The structure of the ionization zones is determined through the assumed value of the mixing length, i.e. by the efficiency of the convective energy transport. Comparing the cases $\alpha = 1$ and $\alpha = 2$, it is seen, that for large efficiency ($\alpha = 2$) relatively small radii are derived, and the difference is seen to increase with decreasing effective temperature. The difference in derived spectral class for models with $\alpha = 1$ and $\alpha = 2$ is negligible for spectral types earlier than about F0, but amounts to several subclasses near G0.

Both the form and the length of the evolutionary tracks in the ($\log T_e$, M_{bol}) diagram show a gradual change in the mass interval which is investigated. Roughly speaking we can say that the models with masses near 1.6 solar masses evolve as heavy main-sequence stars, while the model with 1.0 solar mass evolves as a low mass star. The main-sequence phase of the

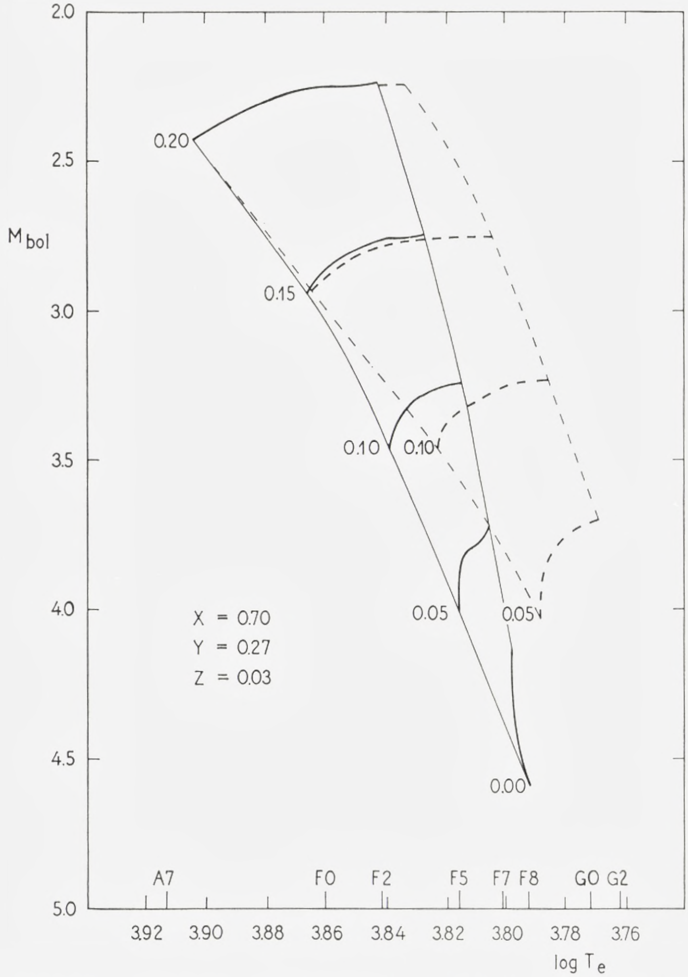

Figure 1. Evolutionary tracks for models with $X = 0.70$, $Y = 0.27$, and $Z = 0.03$ in the ($\log T_e$, $M_{\rm bol}$) diagram. Full drawn curves correspond to models with $\alpha = 2$, while the dashed curves are for $\alpha = 1$. The zero-age line and the upper boundary of the main-sequence band (corresponding to $X_c \simeq 0.10$) are indicated by thin lines. The number attached to each sequence gives the logarithm of the mass in solar units.

model with 1.0 solar mass is characterized by an increase in luminosity with practically constant effective temperature, while the more massive models show a considerable decrease in effective temperature accompanied by a relatively small increase in luminosity. Similar results have been described by Hallgren (1967) and Iben (1967) for Population I models in the same mass range.

Table 1.

Evolutionary tracks for three groups of model sequences. The values of $B-V$ and M_V are derived using the bolometric corrections and the temperature scale published by Harris (1963).

log $M/M_{\odot}$	$X = 0.70$, Age (10^9 years)	$Y = 0.27$, M_{bol}	$Z = 0.03$, log T_e	$\alpha = 1.0$ $B-V$	M_V
0.05	0.00	4.03	3.788	0.54	4.08
	0.30	4.01	3.788	0.54	4.06
	0.70	3.97	3.788	0.54	4.02
	1.10	3.92	3.788	0.54	3.97
	1.50	3.88	3.787	0.55	3.93
	1.90	3.84	3.786	0.55	3.89
	2.70	3.76	3.781	0.57	3.82
	3.00	3.74	3.778	0.58	3.80
	3.30	3.72	3.774	0.59	3.78
	3.50	3.71	3.771	0.60	3.77
	3.60	3.71	3.770	0.60	3.77
	3.70	3.70	3.768	0.61	3.77
0.10	0.00	3.46	3.823	0.43	3.51
	0.50	3.39	3.822	0.44	3.44
	0.80	3.36	3.819	0.44	3.40
	1.10	3.34	3.815	0.45	3.38
	1.30	3.32	3.812	0.46	3.36
	1.50	3.30	3.809	0.47	3.34
	1.70	3.28	3.807	0.48	3.32
	1.90	3.26	3.802	0.50	3.30
	2.05	3.25	3.799	0.50	3.30
	2.20	3.24	3.795	0.52	3.29
	2.35	3.24	3.791	0.53	3.29
	2.45	3.23	3.787	0.55	3.28
	2.55	3.24	3.784	0.56	3.30
0.15	0.00	2.93	3.864	0.32	3.00
	0.10	2.93	3.863	0.32	3.00
	0.30	2.91	3.860	0.33	2.98
	0.50	2.87	3.857	0.34	2.95
	0.90	2.81	3.849	0.35	2.87
	1.10	2.79	3.843	0.37	2.85
	1.30	2.77	3.834	0.39	2.82
	1.40	2.76	3.830	0.41	2.81
	1.50	2.75	3.825	0.42	2.80
	1.60	2.75	3.819	0.43	2.79

(*to be continued*)

TABLE 1 (continued).

$\log M/M_\odot$	$X = 0.70$, Age (10^9 years)	$Y = 0.27$, M_{bol}	$Z = 0.03$, $\log T_e$	$\alpha = 1.0$ $B - V$	M_V
	1.70	2.75	3.812	0.46	2.79
	1.80	2.74	3.805	0.49	2.78
0.20	0.00	2.43	3.904	0.22	2.54
	0.10	2.41	3.902	0.22	2.52
	0.20	2.39	3.900	0.23	2.50
	0.30	2.38	3.896	0.24	2.48
	0.45	2.35	3.891	0.25	2.45
	0.60	2.33	3.886	0.26	2.42
	0.75	2.30	3.880	0.28	2.38
	0.85	2.29	3.873	0.29	2.37
	0.95	2.27	3.868	0.31	2.34
	1.05	2.24	3.862	0.33	2.31
	1.15	2.25	3.850	0.36	2.31
	1.21	2.25	3.844	0.37	2.31
	1.26	2.24	3.839	0.38	2.29
	1.31	2.25	3.832	0.40	2.30

$\log M/M_\odot$	$X = 0.70$, Age (10^9 years)	$Y = 0.27$, M_{bol}	$Z = 0.03$, $\log T_e$	$\alpha = 2.0$ $B - V$	M_V
0.00	0.00	4.59	3.792	0.53	4.64
	0.60	4.54	3.793	0.53	4.59
	1.20	4.51	3.794	0.52	4.56
	1.80	4.46	3.795	0.52	4.51
	2.40	4.40	3.796	0.52	4.45
	3.00	4,34	3.797	0.51	4.39
	3.30	4.32	3.797	0.51	4.37
	3.60	4.29	3.797	0.51	4.34
	3.90	4.26	3.797	0.51	4.31
	4.20	4.22	3.797	0.51	4.27
	4.40	4.19	3.798	0.51	4.24
	4.60	4.16	3.798	0.51	4.21
	4.80	4.14	3.797	0.51	4.19
0.05	0.00	4.01	3.816	0.45	4.05
	0.30	3.98	3.817	0.45	4.02
	0.70	3.95	3.816	0.45	3.99
	1.50	3.87	3.816	0.45	3.91

(to be continued)

TABLE 1 (continued).

$\log M/M_{\odot}$	$X = 0.70$, Age (10^9 years)	$Y = 0.27$, M_{bol}	$Z = 0.03$, $\log T_e$	$\alpha = 2.0$ $B - V$	M_V
	1.90	3.83	3.815	0.45	3.87
	2.20	3.81	3.814	0.46	3.85
	2.50	3.80	3.812	0.46	3.84
	2.80	3.79	3.811	0.47	3.83
	3.00	3.78	3.810	0.47	3.82
	3.20	3.77	3.808	0.48	3.81
	3.40	3.74	3.806	0.48	3.78
	3.50	3.72	3.806	0.48	3.76
0.10	0.00	3.46	3.839	0.38	3.51
	0.20	3.44	3.839	0.38	3.49
	0.50	3.41	3.838	0.39	3.46
	0.80	3.37	3.837	0.39	3.42
	1.10	3.33	3.835	0.40	3.38
	1.40	3.31	3.832	0.40	3.36
	1.60	3.29	3.830	0.41	3.34
	1.80	3.28	3.827	0.42	3.33
	2.00	3.26	3.824	0.43	3.31
	2.20	3.26	3.821	0.44	3.31
	2.40	3.25	3.817	0.45	3.29
	2.55	3.24	3.814	0.46	3.28
	2.60	3.24	3.814	0.46	3.28
	2.65	3.22	3.814	0.46	3.26
0.15	0.00	2.94	3.866	0.31	3.01
	0.10	2.93	3.866	0.31	3.00
	0.30	2.90	3.864	0.32	2.97
	0.50	2.88	3.862	0.32	2.95
	0.70	2.85	3.859	0.33	2.93
	0.90	2.83	3.855	0.34	2.89
	1.10	2.79	3.851	0.35	2.85
	1.30	2.77	3.846	0.37	2.83
	1.50	2.75	3.840	0.38	2.80
	1.60	2.76	3.836	0.39	2.81
	1.70	2.76	3.832	0.40	2.81
	1.80	2.75	3.828	0.41	2.80
	1.85	2.74	3.827	0.42	2.79
0.20	0.00	2.43	3.904	0.22	2.54
	0.10	2.41	3.902	0.22	2.52
	0.30	2.37	3.897	0.24	2.47

(to be continued)

Table 1 (continued).

$\log M/M_{\odot}$	$X = 0.70$, Age (10^9 years)	$Y = 0.27$, M_{bol}	$Z = 0.03$, $\log T_e$	$\alpha = 2.0$ $B - V$	M_V
	0.50	2.35	3.890	0.25	2.44
	0.70	2.31	3.883	0.27	2.40
	0.85	2.28	3.876	0.29	2.36
	1.00	2.26	3.866	0.31	2.33
	1.10	2.26	3.858	0.33	2.33
	1.20	2.25	3.850	0.36	2.31
	1.27	2.25	3.845	0.37	2.31
	1.31	2.24	3.843	0.37	2.30

$\log M/M_{\odot}$	$X = 0.60$, Age (10^9 years)	$Y = 0.37$, M_{bol}	$Z = 0.03$, $\log T_e$	$\alpha = 2.0$ $B - V$	M_V
0.00	0.00	3.70	3.850	0.36	3.76
	0.40	3.66	3.848	0.36	3.72
	0.80	3.61	3.846	0.37	3.67
	1.20	3.57	3.842	0.38	3.63
	1.40	3.55	3.840	0.38	3.61
	1.60	3.53	3.838	0.39	3.58
	1.75	3.52	3.835	0.40	3.57
	1.90	3.51	3.833	0.40	3.56
	2.00	3.50	3.831	0.41	3.55
	2.10	3.48	3.830	0.41	3.53
0.05	0.00	3.20	3.877	0.28	3.28
	0.20	3.18	3.875	0.29	3.26
	0.40	3.15	3.873	0.29	3.23
	0.60	3.11	3.870	0.30	3.19
	0.80	3.07	3.866	0.31	3.14
	0.90	3.07	3.864	0.32	3.14
	1.00	3.06	3.861	0.33	3.13
	1.10	3.05	3.858	0.33	3.12
	1.20	3.05	3.854	0.34	3.11
	1.30	3.04	3.851	0.35	3.10
	1.40	3.03	3.847	0.36	3.09
	1.50	3.02	3.844	0.37	3.08
	1.57	2.99	3.843	0.37	3.05
	1.64	2.97	3.842	0.38	3.03

(to be continued)

TABLE 1 (continued).

$\log M/M_\odot$	$X = 0.60$, Age (10^9 years)	$Y = 0.37$, M_{bol}	$Z = 0.03$, $\log T_e$	$\alpha = 2.0$ $B - V$	M_V
0.10	0.00	2.72	3.912	0.20	2.84
	0.10	2.69	3.911	0.20	2.81
	0.30	2.65	3.906	0.22	2.76
	0.70	2.58	3.891	0.25	2.68
	0.80	2.56	3.887	0.26	2.65
	1.00	2.54	3.872	0.30	2.62
	1.10	2.54	3.863	0.32	2.61
	1.13	2.53	3.862	0.32	2.60
	1.18	2.50	3.859	0.33	2.57
0.15	0.00	2.18	3.952	0.12	2.36
	0.10	2.18	3.947	0.13	2.35
	0.20	2.17	3.942	0.14	2.33
	0.30	2.14	3.937	0.15	2.30
	0.40	2.11	3.932	0.16	2.26
	0.50	2.09	3.925	0.17	2.23
	0.60	2.06	3.918	0.19	2.19
	0.67	2.05	3.910	0.21	2.17
	0.72	2.03	3.906	0.22	2.14
	0.77	2.03	3.900	0.23	2.13
	0.82	2.02	3.894	0.24	2.12
	0.85	2.00	3.892	0.25	2.10
	0.87	1.99	3.891	0.25	2.09
0.20	0.00	1.74	3.982	0.07	1.98
	0.10	1.71	3.977	0.08	1.93
	0.20	1.67	3.971	0.09	1.88
	0.30	1.62	3.965	0.10	1.82
	0.40	1.59	3.955	0.11	1.77
	0.45	1.57	3.949	0.12	1.74
	0.50	1.56	3.943	0.14	1.72
	0.55	1.54	3.935	0.15	1.69
	0.60	1.52	3.926	0.17	1.66
	0.63	1.50	3.921	0.18	1.63
	0.65	1.51	3.917	0.19	1.64

The model with the physical parameters $M = 1.12 M_\odot$, $X = 0.70$, $Z = 0.03$, and $\alpha = 1$ is an interesting intermediate case. The first part of the evolutionary track is very nearly vertical in the diagram, but in the later part of the main-sequence stage the track turns to the right, and this happens before the hydrogen is exhausted in the central region. HALLGREN and DEMARQUE (1966) have recently published a study of the evolution of a star with very nearly the same physical parameters; their results agree well with those of the present investigation. We note especially, that the central convective core, which in the homogeneous model contains 2.6 per cent of the total mass (see Table 2), first grows until it contains slightly more than 4 per

TABLE 2.
Characteristics of three model sequences.

	$X = 0.70$, $Z = 0.03$, $\log M/M_\odot = 0.05$, $\alpha = 1.0$						
Age (10^9 years)	X_c	q_c	f_c	$\log R/R_\odot$	$\log L/L_\odot$	$\log T_c$	$\log \varrho_c$
0.00	0.700	0.026	0.230	0.064	0.230	7.198	1.993
0.30	0.666	0.028	0.253	0.069	0.241	7.204	2.009
0.70	0.619	0.033	0.297	0.077	0.256	7.212	2.032
1.10	0.571	0.036	0.326	0.087	0.277	7.222	2.055
1.50	0.519	0.039	0.344	0.097	0.297	7.231	2.077
1.90	0.457	0.041	0.397	0.107	0.309	7.241	2.110
2.70	0.319	0.042	0.468	0.132	0.339	7.265	2.182
3.00	0.265	0.040	0.497	0.142	0.347	7.275	2.213
3.30	0.194	0.041	0.538	0.154	0.354	7.288	2.257
3.50	0.144	0.039	0.562	0.162	0.358	7.299	2.292
3.60	0.116	0.038	0.561	0.166	0.361	7.305	2.315
3.70	0.087	0.036	0.560	0.170	0.364	7.313	2.344

	$X = 0.70$, $Z = 0.03$, $\log M/M_\odot = 0.20$, $\alpha = 1.0$						
Age (10^9 years)	X_c	q_c	f_c	$\log R/R_\odot$	$\log L/L_\odot$	$\log T_c$	$\log \varrho_c$
0.00	0.700	0.116	0.777	0.153	0.873	7.279	1.883
0.10	0.670	0.115	0.786	0.160	0.881	7.282	1.889
0.20	0.639	0.114	0.795	0.169	0.887	7.285	1.897
0.30	0.606	0.112	0.803	0.178	0.892	7.288	1.905
0.45	0.555	0.105	0.809	0.192	0.902	7.294	1.918
0.60	0.497	0.097	0.816	0.209	0.912	7.300	1.934
0.75	0.433	0.090	0.826	0.228	0.925	7.307	1.952
0.85	0.385	0.084	0.834	0.242	0.928	7.312	1.969
0.95	0.333	0.078	0.842	0.257	0.935	7.319	1.988
1.05	0.276	0.073	0.856	0.275	0.949	7.326	2.009
1.15	0.211	0.067	0.867	0.295	0.943	7.335	2.043
1.21	0.169	0.063	0.875	0.308	0.943	7.342	2.068
1.26	0.131	0.060	0.881	0.319	0.946	7.349	2.093
1.31	0.090	0.056	0.885	0.332	0.945	7.359	2.128

(to be continued)

TABLE 2 (continued).

Age (10^9 years)	X_c	q_c	f_c	$\log R/R_\odot$	$\log L/L_\odot$	$\log T_c$	$\log \varrho_c$
	$X = 0.70,\ Z = 0.03,\ \log M/M_\odot = 0.10,\ \alpha = 2.0$						
0.00	0.700	0.061	0.477	0.074	0.458	7.232	1.971
0.20	0.668	0.064	0.502	0.080	0.468	7.237	1.983
0.50	0.618	0.068	0.538	0.089	0.481	7.244	2.001
0.80	0.566	0.071	0.576	0.099	0.495	7.252	2.020
1.10	0.512	0.064	0.576	0.110	0.510	7.260	2.040
1.40	0.445	0.061	0.597	0.121	0.521	7.269	2.067
1.60	0.395	0.058	0.601	0.128	0.526	7.275	2.088
1.80	0.342	0.058	0.631	0.136	0.533	7.283	2.113
2.00	0.284	0.054	0.653	0.145	0.538	7.292	2.141
2.20	0.220	0.052	0.677	0.153	0.540	7.302	2.177
2.40	0.148	0.048	0.699	0.161	0.542	7.315	2.223
2.55	0.088	0.045	0.709	0.169	0.546	7.329	2.273
2.60	0.067	0.043	0.704	0.172	0.549	7.336	2.297
2.65	0.044	0.040	0.674	0.175	0.556	7.345	2.335

cent of the mass, when the hydrogen content at the centre is about 30 per cent, and thereafter decreases again. Although the convective core only contains a few per cent of the mass, a considerable fraction of the total luminosity is generated in the core. From Table 2 this fraction is seen to increase from 0.23 in the homogeneous stage to 0.56 when the central hydrogen content is 0.15. This is, of course, due to the increasing temperature and density in the central region of the model.

It is interesting to note that a small central convective core is found, even though nearly all energy is generated by the *pp* reactions and not by the *CN* cycle. This seems to be due to the fact, that the detailed dependence of ε_{pp} on T gives a larger concentration of the energy generation towards the centre than the approximate law $\varepsilon_{pp} \approx T^4$, which has often been used in computations in the past. Therefore it is more probable that a convective central core will appear when the more realistic expression for ε_{pp} is used.

KELSALL and STRÖMGREN (1966) have calculated evolutionary tracks of models in the mass range $0.25 \leq \log M/M_\odot \leq 0.85$ for the chemical compositions investigated here. Comparing the zero-age lines derived by KELSALL and STRÖMGREN with the present ones, we find that ours are placed somewhat below those given by KELSALL and STRÖMGREN, the difference being $0\overset{m}{.}10$ for the chemical composition $X = 0.70$, $Z = 0.03$ and $0\overset{m}{.}25$ for $X = 0.60$, $Z = 0.03$. Also from the derived effective temperature

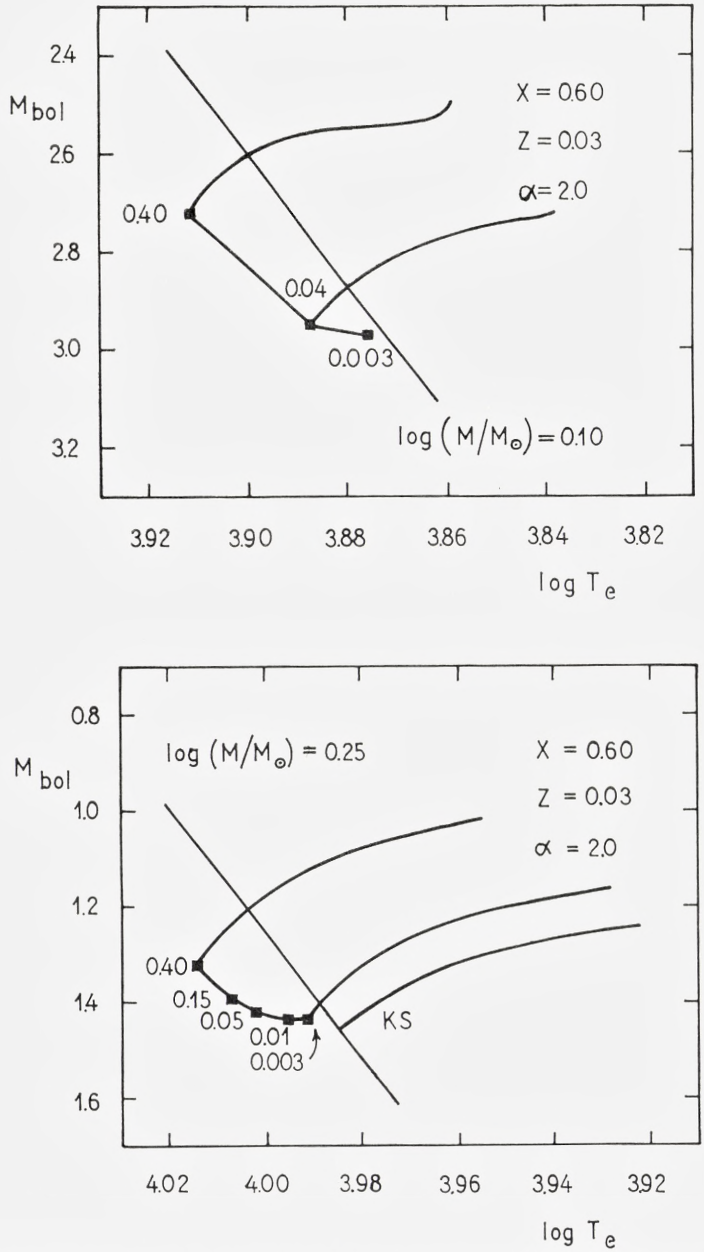

Figure 2–3. Evolutionary tracks of models with outer zones containing different fractions of the total mass. The numbers give the relative mass of the envelopes ($1-q_e$) with an opacity computed without the line contribution. The zero-age lines and the curve marked KS in Figure 2 are taken from Kelsall and Strömgren (1966). Both the luminosity and the effective temperature are seen to decrease with decreasing relative mass of the envelope. The absolute magnitudes given by Kelsall and Strömgren have been corrected by -0.10 (c. f. Section 2).

TABLE 3.
Calibration of the main-sequence band for three groups of model sequences.

	$X = 0.70$,	$Y = 0.27$,	$Z = 0.03$,	$\alpha = 1.0$	
$\log M/M_\odot$	$\log T_e$	log Age	X_c	M_{bol}	ΔM_{bol}
0.05	3.785	9.331	0.423	3.82	0.26
	3.780	9.448	0.300	3.75	0.42
	3.775	9.509	0.214	3.73	0.53
	3.770	8.813	0.128	3.71	0.63
0.10	3.820	8.813	0.556	3.34	0.16
	3.815	9.047	0.500	3.33	0.24
	3.810	9.165	0.431	3.30	0.35
	3.805	9.252	0.347	3.27	0.47
	3.800	9.300	0.286	3.25	0.57
	3.795	9.340	0.225	3.24	0.66
	3.790	9.375	0.160	3.23	0.76
	3.785	9.402	0.099	3.23	0.85
0.15	3.860	8.527	0.641	2.91	0.07
	3.855	8.840	0.551	2.84	0.20
	3.850	8.961	0.486	2.81	0.30
	3.845	9.024	0.434	2.79	0.39
	3.840	9.071	0.388	2.78	0.46
	3.835	9.110	0.347	2.77	0.54
	3.830	9.145	0.304	2.76	0.61
	3.825	9.178	0.256	2.75	0.69
	3.820	9.200	0.220	2.74	0.76
	3.815	9.219	0.185	2.75	0.83
	3.810	9.237	0.147	2.75	0.91
0.20	3.900	8.350	0.643	2.39	0.09
	3.895	8.524	0.595	2.37	0.17
	3.890	8.694	0.540	2.35	0.26
	3.885	8.790	0.490	2.32	0.34
	3.880	8.870	0.437	2.30	0.43
	3.875	8.919	0.397	2.29	0.50
	3.870	8.956	0.357	2.28	0.57
	3.865	8.998	0.308	2.26	0.66
	3.860	9.031	0.263	2.23	0.75
	3.855	9.051	0.231	2.23	0.81
	3.850	9.061	0.210	2.25	0.86
	3.845	9.079	0.177	2.25	0.92
	3.840	9.096	0.140	2.24	1.00
	3.835	9.111	0.106	2.24	1.06

(to be continued)

TABLE 3 (continued).

$\log M/M_\odot$	$X = 0.70$, $\log T_e$	$Y = 0.27$, log Age	$Z = 0.03$, X_c	$\alpha = 2.0$ M_{bol}	ΔM_{bol}
0.05	3.815	9.288	0.437	3.82	0.21
	3.810	9.471	0.243	3.78	0.37
0.10	3.835	9.030	0.519	3.34	0.22
	3.830	9.200	0.399	3.29	0.38
	3.825	9.292	0.297	3.26	0.53
	3.820	9.348	0.210	3.26	0.66
	3.815	9.399	0.106	3.25	0.79
0.15	3.865	8.326	0.666	2.92	0.04
	3.860	8.824	0.556	2.86	0.18
	3.855	8.954	0.484	2.83	0.30
	3.850	9.061	0.404	2.79	0.43
	3.845	9.131	0.330	2.76	0.56
	3.840	9.176	0.267	2.75	0.69
	3.835	9.209	0.213	2.76	0.80
	3.830	9.243	0.147	2.75	0.92
0.20	3.900	8.146	0.638	2.38	0.10
	3.895	8.587	0.586	2.36	0.19
	3.890	8.699	0.538	2.35	0.27
	3.885	8.807	0.484	2.33	0.36
	3.880	8.884	0.431	2.30	0.46
	3.875	8.937	0.383	2.28	0.54
	3.870	8.979	0.339	2.26	0.62
	3.865	9.009	0.299	2.26	0.70
	3.860	9.033	0.264	2.26	0.78
	3.855	9.055	0.227	2.26	0.87
	3.850	9.081	0.181	2.25	0.97
	3.845	9.103	0.133	2.24	1.08

$\log M/M_\odot$	$X = 0.60$, $\log T_e$	$Y = 0.37$, log Age	$Z = 0.03$, X_c	$\alpha = 2.0$ M_{bol}	ΔM_{bol}
0.00	3.845	8.971	0.445	3.60	0.21
	3.840	9.146	0.341	3.55	0.37
	3.835	9.244	0.250	3.52	0.50
	3.830	9.316	0.156	3.49	0.64
0.05	3.875	8.229	0.562	3.18	0.05
	3.870	8.774	0.465	3.11	0.20

(to be continued)

TABLE 3 (continued).

$\log M/M_{\odot}$	$X = 0.60$, $\log T_e$	$Y = 0.37$, log Age	$Z = 0.03$, X_c	$\alpha = 2.0$ M_{bol}	ΔM_{bol}
	3.865	8.936	0.389	3.06	0.32
	3.860	9.013	0.336	3.06	0.42
	3.855	9.073	0.283	3.05	0.54
	3.850	9.124	0.228	3.04	0.66
	3.845	9.167	0.170	3.02	0.78
0.10	3.910	8.176	0.555	2.68	0.07
	3.905	8.519	0.496	2.64	0.17
	3.900	8.690	0.441	2.61	0.27
	3.895	8.792	0.396	2.59	0.36
	3.890	8.860	0.355	2.58	0.44
	3.885	8.922	0.305	2.55	0.53
	3.880	8.966	0.264	2.54	0.62
	3.875	8.993	0.233	2.54	0.69
	3.870	9.010	0.205	2.54	0.77
	3.865	9.035	0.171	2.54	0.85
	3.860	9.065	0.122	2.51	0.96
0.15	3.945	8.169	0.548	2.17	0.09
	3.940	8.370	0.505	2.16	0.18
	3.935	8.529	0.456	2.13	0.28
	3.930	8.638	0.407	2.10	0.37
	3.925	8.704	0.366	2.08	0.46
	3.920	8.759	0.325	2.06	0.55
	3.915	8.800	0.287	2.05	0.63
	3.910	8.829	0.255	2.04	0.70
	3.905	8.865	0.211	2.02	0.79
	3.900	8.887	0.180	2.03	0.86
	3.895	8.909	0.145	2.02	0.93
0.20	3.980	7.483	0.580	1.73	0.04
	3.975	8.187	0.530	1.70	0.14
	3.970	8.354	0.478	1.66	0.25
	3.965	8.476	0.427	1.62	0.36
	3.960	8.561	0.384	1.60	0.46
	3.955	8.606	0.347	1.58	0.54
	3.950	8.648	0.312	1.57	0.62
	3.945	8.684	0.276	1.56	0.71
	3.940	8.715	0.241	1.55	0.79
	3.935	8.740	0.208	1.54	0.87
	3.930	8.762	0.176	1.53	0.95
	3.925	8.782	0.144	1.52	1.03
	3.920	8.803	0.103	1.50	1.11

TABLE 4.
Age calibration for three groups of model sequences.

$X = 0.70$,			$Y = 0.27$,		$Z = 0.03$,		$\alpha = 1.0$	
$\log M/M_\odot$	0.05		0.10		0.15		0.20	
log Age	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}
8.30...	..	..	..	..	..	..	3.900	0.09
8.35...	..	..	..	..	..	..	3.899	0.11
8.40...	..	..	..	..	..	..	3.898	0.12
8.45...	..	..	..	..	..	..	3.897	0.14
8.50...	..	..	..	..	3.859	0.09	3.896	0.16
8.55...	..	..	..	..	3.859	0.10	3.894	0.18
8.60...	..	..	..	..	3.858	0.12	3.893	0.20
8.65...	..	..	..	..	3.858	0.13	3.892	0.23
8.70...	..	..	3.822	0.09	3.857	0.15	3.890	0.26
8.75...	..	..	3.821	0.11	3.856	0.17	3.887	0.30
8.80...	..	..	3.820	0.12	3.855	0.20	3.884	0.35
8.85...	..	..	3.819	0.14	3.853	0.23	3.881	0.41
8.90...	..	..	3.819	0.16	3.851	0.27	3.877	0.48
8.95...	3.789	0.08	3.818	0.19	3.849	0.31	3.871	0.56
9.00...	3.789	0.10	3.816	0.21	3.846	0.36	3.865	0.67
9.05...	3.788	0.13	3.815	0.25	3.842	0.43	3.854	0.82
9.10...	3.788	0.15	3.813	0.29	3.836	0.52	3.839	1.01
9.15...	3.787	0.17	3.811	0.34	3.829	0.62	..	..
9.20...	3.787	0.19	3.808	0.40	3.820	0.76	..	..
9.25...	3.786	0.22	3.805	0.47	3.807	0.97	..	..
9.30...	3.785	0.25	3.800	0.57	..	..	..	..
9.35...	3.784	0.30	3.794	0.69	..	..	..	..
9.40...	3.782	0.35	3.785	0.84	..	..	..	..
9.45...	3.780	0.42	..	..	..	..	..	..
9.50...	3.776	0.51	..	..	..	..	..	..
9.55...	3.770	0.63	..	..	..	..	..	..

$X = 0.70$,			$Y = 0.27$,		$Z = 0.03$,		$\alpha = 2.0$	
$\log M/M_\odot$	0.05		0.10		0.15		0.20	
log Age	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}
8.25...	..	..	..	..	..	..	3.900	0.10
8.30...	..	..	..	..	..	..	3.899	0.11
8.35...	..	..	..	..	..	..	3.899	0.12
8.40...	..	..	..	..	..	..	3.898	0.13
8.45...	..	..	..	..	..	..	3.897	0.15
8.50...	..	..	..	..	..	..	3.896	0.16
8.55...	..	..	..	..	3.863	0.09	3.895	0.19
8.60...	..	..	..	..	3.863	0.10	3.893	0.21

(to be continued)

Table 4 (continued).

$X = 0.70$, $Y = 0.27$, $Z = 0.03$, $\alpha = 2.0$

$\log M/M_{\odot}$	0.05		0.10		0.15		0.20	
log Age	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}
8.65...	..	..	..	..	3.862	0.12	3.892	0.24
8.70...	..	..	..	..	3.862	0.13	3.890	0.27
8.75...	..	..	3.838	0.10	3.861	0.15	3.888	0.31
8.80...	..	..	3.837	0.11	3.860	0.18	3.885	0.36
8.85...	..	..	3.837	0.13	3.859	0.21	3.882	0.42
8.90...	..	..	3.837	0.15	3.857	0.25	3.878	0.48
8.95...	..	..	3.836	0.17	3.855	0.29	3.873	0.57
9.00...	..	..	3.835	0.20	3.853	0.35	3.866	0.67
9.05...	..	..	3.835	0.23	3.851	0.42	3.856	0.85
9.10...	3.817	0.10	3.833	0.28	3.847	0.50	3.846	1.06
9.15...	3.816	0.13	3.832	0.33	3.843	0.61	..	..
9.20...	3.816	0.16	3.830	0.38	3.836	0.76	..	..
9.25...	3.815	0.19	3.828	0.45	3.829	0.95	..	..
9.30...	3.815	0.22	3.824	0.55	..	..	..	..
9.35...	3.814	0.26	3.820	0.66	..	..	..	..
9.40...	3.812	0.30	3.815	0.79	..	..	..	..
9.45...	3.811	0.34	..	..	..	..	..	..
9.50...	3.808	0.42	..	..	..	..	..	..

$X = 0.60$, $Y = 0.37$, $Z = 0.03$, $\alpha = 2.0$

$\log M/M_{\odot}$	0.00		0.05		0.10		0.15		0.20	
log Age	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}
8.05 ...	..	..	..	..	..	..	..	..	3.976	0.12
8.10 ...	..	..	..	..	..	..	3.945	0.09	3.975	0.14
8.15 ...	..	..	..	..	..	..	3.944	0.10	3.974	0.16
8.20 ...	..	..	..	..	..	..	3.943	0.12	3.973	0.18
8.25 ...	..	..	..	..	3.909	0.09	3.943	0.13	3.972	0.20
8.30 ...	..	..	..	..	3.908	0.10	3.942	0.15	3.971	0.22
8.35 ...	..	..	..	..	3.908	0.11	3.940	0.17	3.970	0.26
8.40 ...	..	..	..	..	3.907	0.12	3.939	0.20	3.968	0.30
8.45 ...	..	..	3.874	0.09	3.907	0.14	3.938	0.22	3.966	0.34
8.50 ...	..	..	3.873	0.10	3.906	0.16	3.936	0.26	3.963	0.39
8.55 ...	..	..	3.873	0.11	3.904	0.19	3.934	0.29	3.960	0.46
8.60 ...	3.848	0.09	3.873	0.12	3.902	0.22	3.932	0.34	3.956	0.53
8.65 ...	3.848	0.10	3.872	0.14	3.900	0.25	3.929	0.39	3.950	0.63
8.70 ...	3.847	0.12	3.871	0.16	3.898	0.29	3.925	0.45	3.942	0.75
8.75 ...	3.847	0.13	3.870	0.19	3.896	0.33	3.921	0.54	3.933	0.91
8.80 ...	3.847	0.14	3.896	0.22	3.893	0.38	3.915	0.63	3.921	1.10

(to be continued)

TABLE 4 (continued).

$X = 0.60$,		$Y = 0.37$,		$Z = 0.03$,		$\alpha = 2.0$				
$\log M/M_\odot$	0.00		0.05		0.10		0.15		0.20	
log Age	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}	$\log T_e$	ΔM_{bol}
8.85 ...	3.846	0.16	3.868	0.25	3.891	0.43	3.907	0.75	..	..
8.90 ...	3.846	0.18	3.867	0.29	3.887	0.50	3.897	0.90	..	..
8.95 ...	3.845	0.21	3.864	0.34	3.881	0.60	..	..	..	..
9.00 ...	3.844	0.24	3.861	0.40	3.872	0.73	..	..	..	..
9.05 ...	3.843	0.28	3.857	0.49	3.862	0.91	..	..	..	..
9.10 ...	3.842	0.32	3.852	0.60	..	..	..	..	..	..
9.15 ...	3.840	0.37	3.847	0.73	..	..	..	..	..	..
9.20 ...	3.838	0.44	3.843	0.88	..	..	..	..	..	..
9.25 ...	3.835	0.51	..	..	..	..	..	..	..	..
9.30 ...	3.831	0.61	..	..	..	..	..	..	..	..

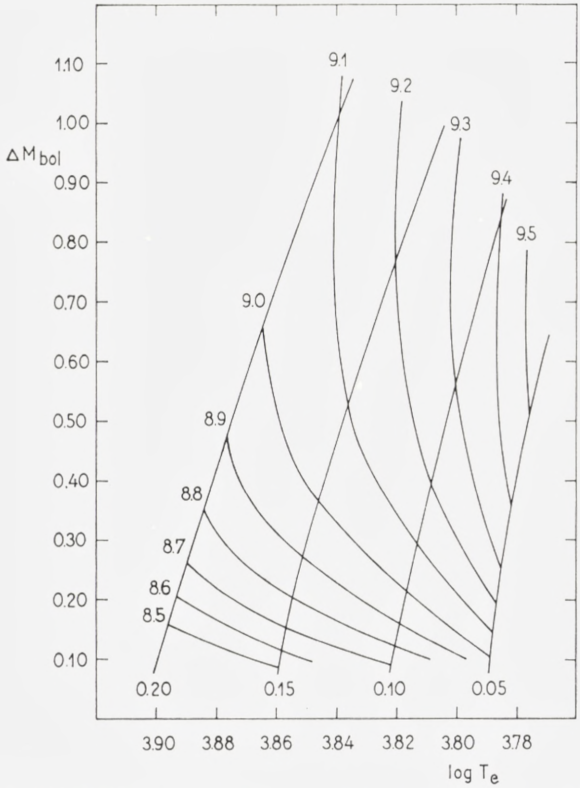

Figure 4. Mass and age calibration for $X = 0.70$, $Y = 0.27$, $Z = 0.03$, and $\alpha = 1$.

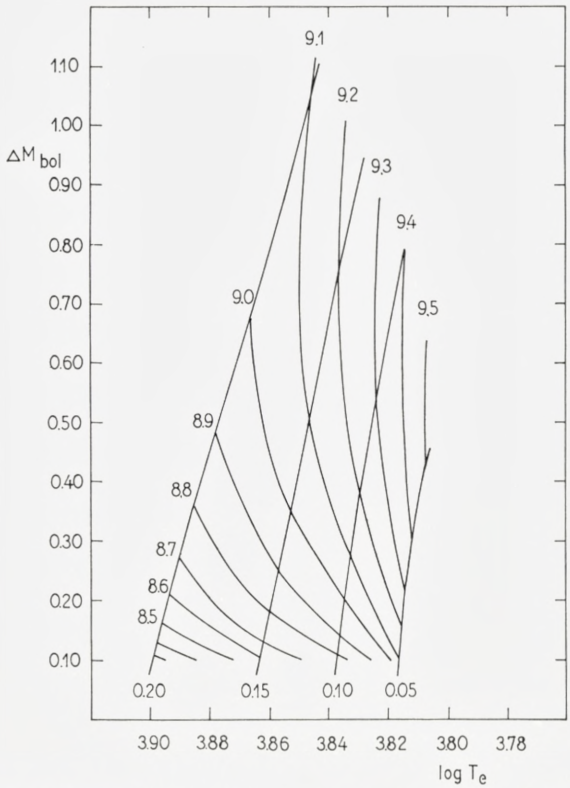

Figure 5. Mass and age calibration for $X = 0.70$, $Y = 0.27$, $Z = 0.03$, and $\alpha = 2$.

and the luminosity of the individual models it is evident that the present models can not be fitted directly to the sequence of models obtained by Kelsall and Strömgren. If we estimate, where we should expect an evolutionary track for a model of a mass given by $\log M/M_{\odot} = 0.25$ to be located, by a rough extrapolation of the results derived for smaller masses, and make a comparison with the corresponding track as given by Kelsall and Strömgren, we find that their models are somewhat less luminous than ours. This is probably due to the fact that different opacities have been used in the envelopes of the models in the two cases. The envelope opacities in our models (based on the Baker-tables) do not include line opacities, and are therefore considerably smaller than those used by Kelsall and Strömgren. We should expect, that the use of smaller envelope opacities would result in larger luminosity and higher effective temperature, and this is precisely the effect which is found.

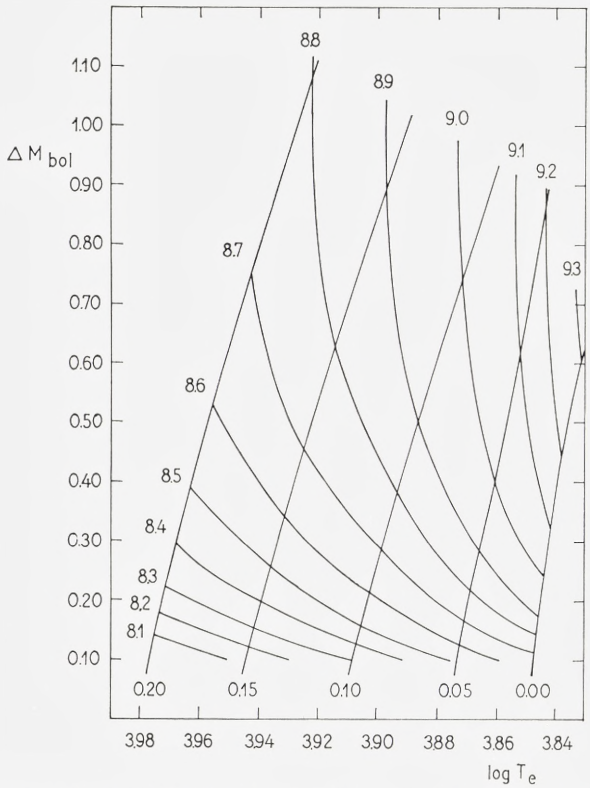

Figure 6. Mass and age calibration for $X = 0.60$, $Y = 0.37$, $Z = 0.03$, and $\alpha = 2$.

In order to investigate this matter in more detail we have constructed similar models in which the envelopes contain a much smaller part of the total mass, by means of the computer programmes described by PETERSEN (1966). The results are shown in Figures 2 and 3. The number attached to each sequence gives the relative mass of the envelopes with an opacity computed without the line contribution. It is seen that the luminosity does increase when the relative mass of the envelope increases, as was to be expected.

The calibration of the HERTZSPRUNG-RUSSELL diagram has been performed by means of second order interpolation in the data of the model sequences just described. Following KELSALL and STRÖMGREN (1966) we have, for equidistant values of $\log T_e$, computed $\Delta M_{\rm bol}$, defined as $M_{\rm bol}$ (zero-age line) $-$ $M_{\rm bol}$ (model sequence), using the zero-age models of the chemical composition in question, to derive $M_{\rm bol}$ for the zero-age line. In order to facilitate the age calibration we have also calculated $\log T_e$ and ΔM_{bol} corresponding to the age values given by log (Age in years) equal to 8.05, 8.10, ... up to the last model in the main-sequence stage ($X_c \simeq 0.08$) for each evolutionary track. The results are given in Tables 3 and 4, and curves of equal mass and equal age in the ($\log T_e$, ΔM_{bol}) diagram are shown in Figures 4, 5, and 6. From such figures the age and mass of a single main-sequence star can be derived, provided that observations can supply sufficiently accurate information about $\log T_e$, ΔM_{bol}, and the chemical composition of the star.

4. Discussion

Comparing Figures 4, 5, and 6 with the calibration diagrams for B and A stars in the paper by Kelsall and Strömgren (loc. cit.) it is seen that the interval in ΔM_{bol} covered by a model sequence (curve of constant mass) becomes smaller for the later spectral types. This expresses the fact that the expected accuracy in the age determination, by a comparison of observations and the calibration curves, is higher for the early spectral types than for the later. And for spectral types later than about G0 this method for age determination of main-sequence stars clearly can not be applied at all, since the evolutionary tracks in the corresponding region of the Hertzsprung-Russell diagram follow the zero-age line, in contrast to the upper main-sequence, where the tracks are nearly perpendicular to this line.

In addition, also other effects tend to make the age determination for the later spectral types more difficult than for earlier types. The influence of differences in chemical composition becomes important for the later types because the value of ΔM_{bol}, that is derived from observations of a certain star, is dependent on the chemical composition, characterized by X and Z, determining which zero-age line one has to use. Consequently accurate determinations of mass and age can only be obtained if, for the sample of stars considered, the range of variation of X and Z is fairly limited. With regard to Z, selection of the sample of stars according to observations of a metal-content index may lead to a satisfactory solution. A discussion of this question is beyond the scope of the present investigation since the calculations have been carried out for one Z value only. With regard to the influence of variation in X, we conclude from our calculations that the effect on age determination is serious for the later spectral types unless the range of variation of X is relatively small (less than 0.1), which may however well be the case.

We note in this connection that when ΔM_{bol} is determined spectroscopically, what matters in the context mentioned is the change of the relation between surface gravity and effective temperature with X.

Uncertainty concerning the appropriate value of the mixing length also contributes considerably to the uncertainty in the derived values of mass and age for the later types. From Figures 4 and 5 it is seen that this uncertainty is small for $\log T_e \geq 3.86$, but that it increases quickly for lower effective temperatures. For instance, for $\log T_e = 3.824$ and $\Delta M_{\mathrm{bol}} = 0.54$ we will derive $\log M/M_\odot = 0.13$ and log (Age in years) = 9.16 from Figure 4 (the case of $\alpha = 1$), while Figure 5 ($\alpha = 2$) gives the values 0.10 and 9.30 for the same quantities. Since the value of α is about 1.5 for main-sequence stars (see e.g. BAKER 1963) the realistic values of mass and age are probably intermediate between the values derived from Figures 4 and 5.

For earlier spectral classes, on the other hand, the interval in T_e covered by an evolutionary track is considerably larger and ΔM_{bol}, therefore, does not depend critically on the chemical composition, especially near the upper boundary of the main-sequence.

In view of above mentioned circumstances it seems unlikely that calibration of the present type (for stars with $X_c > 0.08$) can be utilized for accurate age determination for stars of spectral classes later than about F3. This means that the method considered here can only give reasonably accurate results for stars younger than about $2-3 \times 10^9$ years. Ages derived for spectral types later than about F3 will probably have only statistical significance.

It is noticeable that the isochrones in Figures 4, 5, and 6 are almost vertical in the upper part of the main-sequence band. Therefore any measure of effective temperature, for instance a colour index, is also an indicator of age in this part of the diagram. DENNIS (1966) has used this fact to derive ages for three groups of stars with $0.220 \leq \text{b-y} < 0.290$, $0.290 \leq \text{b-y} < 0.330$, and $0.330 \leq \text{b-y} \leq 0.390$, respectively, by means of the present age-calibration. He found, as would be expected, that the dispersion in the velocity perpendicular to the plane of the Galaxy for these three groups of stars increases with the derived ages.

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